Lagrange multipliers

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Last time: chain rule, maxima/minima

Upshot

1. critical point iff gradient vanishes
2. local max/min $\Rightarrow$ critical point but not conversely
3. in two-dimensions, can use Hessian determinant:

$$\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2$$

$$\underbrace{\text{Parametric vs Equation form of a curve in } \mathbb{R}^2}$$

A curve, usually denoted by γ is a 1-D subset of $\mathbb{R}^2$

Given by either

- Parametric: $(x, y) = (x(t), y(t))$
  $t \in \mathbb{R}$
- Equation: $g(x, y) = 0$
  then $\gamma = \{(x, y) \in \mathbb{R}^2 \mid g(x, y) = 0\}$

eg a circle

parametric $(x, y) = (\cos t, \sin t)$

eqn $x^2 + y^2 - 1 = 0$

What if we want a parametric form in which one of the variables is the parameter?

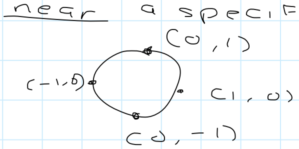
ie. $(x, y) = (t, y(t))$

or $(x, y) = (x(t), t)$

In 1st case, eqn form is $y - y(x) = 0$

eg for a circle

- say $x = t$
  get $y = \sqrt{1 - x^2}$
  (or $y = -\sqrt{1 - x^2}$)
- can't find single eqn for y in terms of x everywhere (globally)
- but can often find it locally
  ie near a specific point



eg near $(0, 1)$ have local parametric form

$y = -\sqrt{1 - x^2}$

Q/ but at $(1, 0)$ what's the sqrt?

A/ can't solve for x in terms of x near $(1, 0)$ or $(-1, 0)$ bc the tangent line is vertical

Similarly near $(0, \pm 1)$, can't locally solve for x in terms of y bc the tangent line is horizontal

Implicit Fcn Thm

Implicit Fcn Thm

Suppose $g(x,y)=0$ describes a curve γ and that $a,b \in \gamma$ and (a,b) is not a critical point of g

Then

① If tan line to γ at (a,b) is not vertical then we can solve for y in terms of x near (a,b).

Precisely:

$\Rightarrow$ can find fcn f defined on a nbhd of a (ie some open interval containing a) such that $(x,y) = (t, f(t))$ describes the curve γ near (a,b)

② If the tangent line is not horizontal at (a,b), can solve for x in terms of y near (a,b)

Caveat:

Only works as long as (a,b) is not a critical point of g

Q/ When is tangent line vertical

A/ Recall tangent line is given by

$$\frac{\partial g}{\partial x}(a,b)\cdot(x-a) + \frac{\partial g}{\partial y}(a,b)\cdot(y-b) = 0$$

this is vertical if $\frac{\partial g}{\partial y}(a,b) = 0$

Better Formulation of Implicit Fcn thm

① If $\frac{\partial g}{\partial y}(a,b) \neq 0$ then can solve for y in terms of x near (a,b)

② If $\frac{\partial g}{\partial x}$ then can solve for x in terms of y near (a,b)

Notice thm automatically doesn't apply if (a,b) is a critical point of g

$\Rightarrow$ we don't have to explicitly require that (a,b) is not critical

Generalizations

- In $\mathbb{R}^3$ consider $g(x_1,x_2,x_3)=0$. This defines a surface (not a curve).
- If at (a_1,a_2,a_3) we have $\frac{\partial g}{\partial x_i} \neq 0$ then can solve for x_i in terms of the other two variables near (a_1,a_2,a_3)
- Similar in $\mathbb{R}^n$. Then $g(x_1,\dots,x_n)=0$ defines an $(n-1)$ dimensional subset of $\mathbb{R}^n$ and there are $n-1$ parameters.
- What about a curve in $\mathbb{R}^3$?

  Then, need to consider $g(x_1,x_2,x_3) = (g_1, g_3)$

  ie. need to solve $g(x_1,x_2,x_3) = (0,0)$

  for $g: D \to \mathbb{R}^2$, $D \subset \mathbb{R}^3$

  now you have a 2×3 matrix and you consider determinants of 2×2

eg can solve for x_2 and x_3 in terms of x_1 if:

$$\det \begin{vmatrix} \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \end{vmatrix} \neq 0$$

at the given

use $q \times q$ minors of $q \times n$ matrix

Lagrange Mult

Suppose f a fcn in $\mathbb{R}^2$ and γ is a curve.

Suppose we want to maximize (or minimize) $f(x,y)$ among $(x,y) \in \gamma$

How?

if we have a parametrization
$(x,y) = (x(t), y(t))$
of γ then its easy using chain rule.

Why? just need to solve $\frac{df}{dt} = 0$

Notice $\frac{df}{dt} = \frac{d f(x(t), y(t))}{dt} = \nabla f \cdot (x'(t), y'(t))$

use chain rule for composition

$$\mathbb{R} \xrightarrow{(x(t),\, y(t))} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$$

so $\frac{df}{dt} = 0$ precisely when $(x'(t), y'(t))$ aka tangent vector to γ, is $\perp$ to ∇f.

But what if we don't have a parameterization?
(if γ given by $g(x,y) = 0$?)

one attempt at an answer.
- if (x,y) isn't a critical point, the implicit fcn thm says there is a parameterization
  BUT doesn't say how to compute it

Lagrange's idea: use ∇g instead of $(x'(t), y'(t))$

How? Tangent line at (a,b) is given by

$$\nabla g(a,b) \cdot [(x,y) - (a,b)] = 0$$

$\Rightarrow$ the tangent vector is $\perp$ to ∇g
ie, for any parameterization $(x(t), y(t))$ of γ we have
$\nabla g \cdot (x'(t), y'(t)) = 0$

therefore, $\nabla f \perp (x'(t), y'(t))$ iff $\nabla f \parallel \nabla g$

in summary
- ∇g always $\perp$ tangent vector
- ∇f $\perp$ tangent vector whenever $\frac{df}{dt} = 0$
- $\nabla f \parallel \nabla g$ when $\frac{df}{}$

∇g always $\perp$ tangent vector
∇F $\perp$ tangent vector whenever $\frac{dF}{dt} = 0$
$\nabla F \parallel \nabla g$ when $\frac{dF}{dt} = 0$

$\rightarrow$ (and this condition doesn't refer to the parameterization)

When is $\nabla F \parallel \nabla g$?

<u>if</u>: $\nabla F = \lambda \nabla g$ for $\lambda \in \mathbb{R}$

<u>note</u>: $\nabla F = \lambda \nabla g$

$\iff \frac{\partial F}{\partial x} = \lambda \frac{\partial g}{\partial x} \iff \frac{\partial F}{\partial y} = \lambda \frac{\partial g}{\partial y}$

solve these 2 eqns along with the third eqn:

$g(x, y) = 0$

for x, y, λ

<u>Note</u> can do same in 3-dim

$\rightarrow$ get 4 eqns for the 4 vars x, y, z, λ

bc $\nabla F, \nabla g \perp$ to tangent plane

Double Integrals

Q given $F(x,y)$ what does it mean to integrate F?

<u>idea</u>: Partial integral wrt one of the variables and consider the other variable as a const

<u>eg</u>

$F(x, y) = x^2 y$

$\int F\,dx = \frac{yx^3}{3} + C$

$\int F\,dy = \frac{x^2 y^2}{2} + C$

gives some notion of indefinite integral

Q/What about definite integral?

$$\int_1^2 F\,dx = \frac{yx^3}{3}\Big|_{x=1}^{x=2} = \frac{y(2)^3}{3} - \frac{y(1)^3}{3} = \frac{7y}{3}$$

<u>Notice</u>: still fcn of y.

<u>Similarly</u>:

$$\int_1^2 F\,dy = \left[x^2 y^2 / 2\right]_{y=1}^{y=2} = \frac{4x^2}{2} - \frac{x^2}{2} = \frac{3x^2}{2}$$

Q/ How to get # as a definite integral?

A/ integrate twice, once wrt each var

<u>eg</u>

$$\int_1^2 \left[\int_1^2 F\,dx\right] dy = \int_1^2 \frac{7y}{3}\,dy = \left[\frac{7y^2}{6}\right]_1^2 = \frac{28}{6} - \frac{7}{6} = \frac{21}{6} = \frac{7}{2}$$

Let's try:

$$\int_1^2 \left[\int_1^2 F\,dy\right] dx = \int_1^2 \frac{3x^2}{2}\,dx = \left[\frac{x^3}{2}\right]_{x=1}^{x=2} = \frac{8}{2} - \frac{1}{2} = \frac{7}{2}$$

this is

$$\int_{y=1}^{y=2}\int_{x=1}^{x=2} f\,dx\,dy$$

instead we could go from $x=2$ to $x=3$ but still $y=1$ to $y=2$

then we get.

$$\int_{y=1}^{y=2}\int_{x=2}^{x=3} f\,dx\,dy = \int_{x=2}^{x=3}\left[\frac{3x^2}{2}\right]dx = \left[\frac{x^3}{2}\right] = \frac{27}{2} - \frac{8}{2} = \frac{19}{2}$$